ON NUMERICAL METHODS FOR THE TREATMENT OF HYDRODYNAMIC SHOCKS

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THESIS

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ABSTRACT

Several numerical methods used in the calculation of hydrodynamic shocks were investigated. Particular attention was given to the artificial viscosity approach of Von Neumann and Richtmyer and its application to the "PUFF" numerical scheme. The particle model approach of Ludford, Polachek and Seeger, and the method of Lax were also considered.

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I. INTRODUCTION

Frequently the numerical calculation of compressible fluid flow is complicated by the presence of shock waves. The difficulties arise from the fact that shock waves propagate discontinuities in velocity, pressure, and other variables characterizing the fluid flow. Various approaches to this problem have been offered, each having its own desirable and undesirable characteristics. These approaches generally fall into two categories. The first category is related to the study of the viscosity of the fluid, whereas the second category is dependent on the conservation form of the hydrodynamic equations. The first category of approaches will receive primary attention.

II. NUMERICAL CALCULATION OF HYDRODYNAMIC SHOCKS USING AN ARTIFICIAL VISCOSITY FACTOR

A. INTRODUCTION

The equations describing perfect compressible fluid flow, in the presence of shock waves, produce solutions with discontinuities. Investigation of the physical situation shows that the true discontinuities, however, cannot occur due to the viscosity, or inner friction, of the fluid. These equations ignore the viscosity of the fluid and do not accurately represent the physical system. The addition of viscosity terms to this system of equations shows that the fluid behavior inside the shock region is nonlinear but continuous. The viscosity of the fluid is negligible outside of the shock and significant inside the shock. The original intent then, was to replace the shock region by a discontinuity and treat the shock as a two-sided boundary. The size of the jump discontinuity, or rather the boundary values, would be prescribed by the Rankine-Hugoniot equations. However this approach has several drawbacks. First, the presence of a discontinuity complicates the use of a numerical scheme to solve the problem. Secondly, the shock wave is in motion, and hence, the boundary is moving. Thirdly, since irreversible thermodynamic changes of state take place across a shock region, an increase in the specific entropy of the fluid must be added to the original jump conditions. And, lastly, this approach does not represent the physical situation in that there is no indication of the behavior of the fluid inside the shock region.

Since the addition of the true viscosity terms severely complicates the system of differential equations, Von Neumann and Richtmyer [Ref. 7] have suggested the addition of a "pseudoviscosity" term to the equations of non-viscous flow, which will not complicate the system as much as the true term would. This term must, of course, conform to several restrictions.

B. THE BASIC EQUATIONS

The equations describing one-dimensional flow of a compressible fluid are as follows:

$$V(x,t) = \frac{1}{\rho_0(x)} \frac{\partial X}{\partial x} \tag{2.1}$$

$$U(x,t) = \frac{\partial X}{\partial t} \tag{2.2}$$

$$\rho_0 \frac{\partial U}{\partial t} = - \frac{\partial (p+q)}{\partial t} \tag{2.3}$$

$$\frac{\partial E}{\partial t} + (p+q) \frac{\partial V}{\partial t} = 0 \tag{2.4}$$

$$\rho_0 \frac{\partial V}{\partial t} = \frac{\partial U}{\partial x} \tag{2.5}$$

$$E = \frac{pV}{(\gamma-1)} \tag{2.6}$$

where x is the Lagrangean coordinate, X = X(x,t) is the Eulerian coordinate (i.e., X(x,t) gives the position, at time t, of a fluid element initially at position x), $\rho_0(x)$ is the initial density, V is the specific volume, U is the fluid velocity, p is the static fluid pressure, E is the internal energy per unit mass, γ is the ratio of specific heats (i.e., c_p / c_v), and q is the artificial viscosity. Equations (2.3), (2.4), and (2.5) are the equations of motion, of energy, and of continuity respectively. Equation (2.6) is the equation of state for a perfect gas.

C. THE ARTIFICIAL VISCOSITY FACTOR

The expression for q must satisfy the following requirements:

1. Equations (2.3), (2.4), and (2.5) must possess solutions without discontinuities.
2. The thickness of the shock must be everywhere of the same order as Δx (the length increment) used in the numerical calculation.
3. The effect of q on eqns. (2.3) and (2.4) must be negligible outside the shock region.
4. As $\Delta x \to 0$, the solution must approach a state with a jump discontinuity prescribed by the Rankine-Hugoniot equations.

Apparently, these requirements are not enough to uniquely define q.

In any case the expression that Von Neumann and Richtmyer developed is

$$q = - \frac{(\rho_0 c \Delta x)^2}{V} \frac{\partial V}{\partial t} \left| \frac{\partial V}{\partial t} \right| . \tag{2.7}$$

where c is a dimensionless constant near unity. By the use of equation (2.5) q can be written as

$$q = - \frac{(c \Delta x)^2}{V} \frac{\partial U}{\partial x} \cdot \left| \frac{\partial U}{\partial x} \right| . \tag{2.8}$$

Von Neumann and Richtmyer have proven that q satisfies the above requirements for a particular case of steady-state plane shock. They conjecture, however, that the artificial viscosity approach would be equally suited to more complicated multi-dimensional flows. The problem they consider is the example of a one-dimensional shock wave separating two regions of constant state. This simulates the situation that occurs when a piston is pushed at a constant velocity into a long tube containing a fluid initially at rest. After the shock has traveled a sufficient distance from the initiating piston, it moves at a constant speed, s. In the absence of an artificial viscosity term, the specific volume, V, at some time t, is given by figure 1. Since we are considering steady-state solutions only, the solutions depend only on a linear combination of x and t given by

$$w = x - st. \tag{2.9}$$

Define

$$M = \rho_0 s. \tag{2.10}$$

Now

$$U(x,t) \to U(w) = U(x - st)$$

implies that equation (2.3) becomes

$$M \frac{dU}{dw} = \frac{d(p + q)}{dw} \tag{2.11}$$

since

$$M \frac{\partial U}{\partial w} = \rho_0 s \frac{\partial U}{\partial w} = \rho_0 \frac{\partial w}{\partial t} \frac{\partial U}{\partial w} = \rho_0 \frac{\partial U}{\partial t}$$

$$= - \frac{\partial (p+q)}{\partial x} = \frac{d(p+q)}{dw} \frac{\partial w}{\partial x} = \frac{d(p+q)}{dw} .$$

Similarly, equations (2.4) and (2.5) become

$$\frac{dE}{dw} + (p+q) \frac{dV}{dw} = 0 \tag{2.12}$$

and

$$-M \frac{dV}{dw} = \frac{dU}{dw} \tag{2.13}$$

Then equations (2.11) and (2.13) give

$$-M^2 \frac{dV}{dw} = \frac{d(p+q)}{dw} \tag{2.14}$$

and equations (2.12) and (2.14) give

$$\frac{dE}{dw} + \frac{d[(p+q)V]}{dw} + M^2 V \frac{dV}{dw} = 0 \tag{2.15}$$

Von Neumann and Richtmyer then integrated equations (2.13), (2.14), and (2.15) with respect to w giving

$$MV + U = C_1 \tag{2.16}$$

$$M^2V + p+q = C_2 \tag{2.17}$$

$$E + (p+q)V + 1/2\ M^2V^2 = C_3 \tag{2.18}$$

as solutions of equations (2.13), (2.14), and (2.15) where C_1, C_2, and C_3 are constants of integration. Let the initial and final values be denoted by:

$$\text{As } w\to\infty;\ V\to V_i,\ p\to p_i,\ E\to E_i,\ q\to 0 \tag{2.19}$$

$$\text{As } w\to-\infty;\ V\to V_f,\ p\to p_f,\ E\to E_f,\ q\to 0 \tag{2.20}$$

Since V_i and V_f are particular values of V, and p_i, and P_f are particular values of p, they satisfy equation (2.17) giving

$$M^2V_i + p_i + 0 = C_2$$

$$M^2V_f + p_f + 0 = C_2 ,$$

which implies that

$$M^2(V_i-V_f) = (p_f-p_i) . \tag{2.21}$$

A similar argument using equation (2.18) yields

$$E_f + p_fV_f + 1/2\ M^2V_f^{\ 2} = C_3$$

$$-E_i - p_iV_i - 1/2\ M^2V_i^{\ 2} = C_3$$

from which it follows that

$$(E_f-E_i) = 1/2\ M^2(V_i^{\ 2}-V_f^{\ 2}) + p_iV_i - p_fV_f .$$

Then by equation (2.21)

$$(E_f - E_i) = 1/2 \frac{p_f - p_i}{V_i - V_f} (V_i^2 - V_f^2) + p_i V_i - p_f V_f$$

$$= 1/2 (p_f - p_i)(V_i + V_f) + p_i V_i - p_f V_f$$

$$= 1/2 (p_i V_i + p_f V_i - p_i V_f - p_f V_f) ,$$

or

$$(E_f - E_i) = 1/2 (p_i + p_f)(V_i - V_f) \qquad (2.22)$$

Von Neumann and Richtmyer point out that equations (2.21) and (2.22) are independent of q, providing $q \to 0$ as $w \to \pm\infty$, and in fact are the Hugoniot equations. Requirement (4) is then satisfied since it has just been shown that the Rankine-Hugoniot conditions are satisfied for the flow sufficiently far from the shock region. Requirement (4) may be examined in an alternate fashion. Let $Z(x,t)$ be the solution to the set of equations describing non-viscous flow. I.e., $Z(x,t)$ has a jump discontinuity, prescribed by the Rankine-Hugoniot equations, in the shock region. Now let $w(x,t,\Delta x)$ be the solution of the equations of viscous flow corresponding to a fixed q, or more accurately, a fixed Δx. Then we require

$$\lim_{\Delta x \to 0} w(x,t,\Delta x) = w(x,t,0) = Z(x,t) \qquad (2.23)$$

By the very fashion in which q was introduced equation (2.23) is satisfied. Note that q actually has the dimensions of pressure and enters equation (2.3) and (2.4) in the forms $\frac{\partial(p+q)}{\partial t}$ and $(p+q)$ respectively. Since q is continuous, $\frac{\partial q}{\partial x} \to 0$ as $q \to 0$, and hence, all viscosity terms approach zero. Consequently, the system of equations describing viscous flow approaches the system describing nonviscous flow as our mesh size approaches zero.

The question naturally arises as to why Von Neumann and Richtmyer have created a process whereby decreasing the mesh size causes $w(x,t,\Delta x)$ to approach a solution that does not represent the physical system. The answer is a heuristic one in that it must be possible to make q arbitrarily small to accomodate arbitrarily thin shock waves. It should also be noted that q is physically artificial and was created for numerical convenience only.

To investigate the shape of the shock Von Neumann and Richtmyer consider solutions satisfying

$$\frac{\partial V}{\partial t} \leq 0, \quad \text{or equivalently,} \quad \frac{\partial V}{\partial w} \geq 0. \tag{2.24}$$

Equations (2.24) are normally the situation characterizing a shock moving to the right. Then equation (2.7) yields

$$qV = (Mc\Delta x)^2 \left(\frac{dV}{dw}\right)^2 . \tag{2.25}$$

Now from equation (2.18)

$$E + 1/2\, M^2V^2 = C_3 - (p+q)V$$

$$E - 1/2\, M^2V^2 = C_3 - (p+q)V - M^2V^2$$

$$= C^3 - V[(p+q) - M^2V].$$

And then equation (2.17) gives

$$E - 1/2\, M^2V^2 = C_3 - C_2V . \tag{2.26}$$

Then by equation (2.6)

$$pV = (E)(\gamma-1)$$

$$= \frac{\gamma-1}{2} M^2V^2 + C_3(\gamma-1) - VC_2(\gamma-1)$$

$$= \left[\frac{\gamma-1}{2} \right] M^2V^2 + C_4V + C_5 \tag{2.27}$$

where

$$C_4 = -C_2\ (\gamma-1) \tag{2.28}$$

$$C_5 = C_3\ (\gamma-1) \tag{2.29}$$

Then equation (2.17) yields

$$qV = C_2V - pV - M^2V^2$$

$$= C_2V - C_4V - C_5 - [\ \tfrac{\gamma-1}{2}\ M^2V^2 + M^2V^2]$$

$$qV = C_2V - [\tfrac{(\gamma+1)}{2}]\ M^2V^2 - C_4V - C_5\ . \tag{2.30}$$

Now for $V = V_i$ and $V = V_f$, $q = 0$. Therefore since the right side of equation (2.30) is quadratic in V and vanishes for $V = V_i$ and $V = V_f$,

$$qV = -[\tfrac{\gamma+1}{2}]\ M^2V^2 + (C_2-C_4)V-C_5$$

$$= \tfrac{\gamma+1}{2}\ M^2(-V+V_i)(V-V_f)\ . \tag{2.31}$$

Then equation (2.25) yields

$$(Mc\Delta x)^2\ [\tfrac{dV}{dw}]^2 = \tfrac{\gamma+1}{2}\ (V_i-V)(V-V_f)$$

$$(c\Delta x)^2\ [\tfrac{dV}{dw}]^2 = \tfrac{\gamma+1}{2}\ (V_i-V)(V-V_f) \tag{2.32}$$

To solve equation (2.32), Von Neumann and Richtmyer proceed as follows: Let,

$$A = V - \frac{V_i+V_f}{2}\ ,\quad A_o = \frac{V_i-V_f}{2}\ ,\quad B = \frac{A}{A_o} \tag{2.33}$$

Then,

$$c\Delta x\ \frac{dV}{dw} = [\tfrac{\gamma+1}{2}]^{1/2}\ (V_i-V)^{1/2}\ (V-V_f)^{1/2}$$

$$c\Delta x \frac{dA}{dw} = [\frac{\gamma+1}{2}]^{1/2} [\frac{V_i-V_f}{2} - A]^{1/2} [A - \frac{V_f-V_i}{2}]^{1/2}$$

$$= [\frac{\gamma+1}{2}]^{1/2} [(A_o-A)(A+A_o)]^{1/2} .$$

Therefore,

$$c\Delta x \frac{dA}{dw} = [\frac{\gamma+1}{2}]^{1/2} [(A_o{}^2-A^2)]^{1/2} , \tag{2.34}$$

or

$$c\Delta x(\frac{1}{A_o}) \frac{dA}{dw} = [\frac{\gamma+1}{2}]^{1/2} [\frac{A_o{}^2-A^2}{A_o{}^2}]^{1/2} , \tag{2.35}$$

giving

$$c\Delta x \frac{dB}{dw} = [\frac{\gamma+1}{2}]^{1/2} [1 - B^2]^{1/2} . \tag{2.36}$$

Hence,

$$dw = [\frac{2}{\gamma+1}]^{1/2} c\Delta x [\frac{dB}{(1-B^2)^{1/2}}]$$

$$w = [\frac{2}{\gamma+1}]^{1/2} c\Delta x \int \frac{dB}{(1-B^2)^{1/2}}$$

$$w = w_o \arcsin B , \tag{2.37}$$

where

$$w_o = [\frac{2}{\gamma+1}]^{1/2} c\Delta x \tag{2.38}$$

Finally,

$$A = A_oB = A_o \sin \frac{w}{w_o} = \frac{V_i-V_f}{2} [\sin \frac{w}{w_o}], \tag{2.39}$$

or

$$V = \frac{V_i - V_f}{2} \sin \frac{w}{w_o} + \frac{V_i + V_f}{2} . \qquad (2.40)$$

V is obviously continuous and hence requirement (1) is satisfied.

Von Neumann and Richtmyer state that w_o is a measure of shock thickness. Thus, if c is near unity, w_o is $O(\Delta x)$ and requirement (2) is satisfied.

Now taking the derivative of V with respect to w and setting it equal to zero gives

$$\frac{dV}{dw} = \frac{V_i - V_f}{2} \frac{1}{w_o} \cos \frac{w}{w_o} = 0 . \qquad (2.41)$$

Hence

$$\cos \frac{w}{w_o} = 0 , \qquad (2.42)$$

giving $w = \frac{(2n+1)\pi}{2} w_o$, (n an integer), as points where V assumes its relative maxima and minima. Similarly,

$$\frac{d^2V}{dw^2} = \left(-\frac{V_i + V_f}{2} \right) \frac{1}{w_o^2} \sin \frac{w}{w_o} = 0 \qquad (2.43)$$

implies

$$\sin \frac{w}{w_o} = 0 , \qquad (2.44)$$

giving $w = n\pi w_o$, (n an integer), as inflection points for V. Now for $w = -\frac{\pi}{2} w_o$,

$$V = \frac{V_i - V_f}{2} \sin \left(-\frac{\pi}{2}\right) + \frac{V_i - V_f}{2} = V_f . \qquad (2.45)$$

For $w = \frac{\pi}{2} w_0$,

$$V = \frac{V_i - V_f}{2} \sin\left(\frac{\pi}{2}\right) + \frac{V_i + V_f}{2} = V_i . \quad (2.46)$$

And for $w = 0$

$$V = \frac{V_i + V_f}{2} \quad (2.47)$$

Finally, since only solutions satisfying $\frac{dV}{dw} \geq 0$ were considered, V is non-oscillatory. As a result of equations (2.41) through (2.47) figure 2 represents our solution.

Note that for this particular problem of steady-state plane shock, $q \equiv 0$ outside $[-\frac{\pi}{2} w_0, \frac{\pi}{2} w_0]$(i.e., the shock layer) since $\frac{\partial V}{\partial t} = 0$ in this region. Normally, outside the shock region, q would be negligible in comparison with the static pressure p because of the factor $(\Delta x)^2$ in equation (2.7) and a relatively small value for $\frac{\partial V}{\partial t}$. However, inside the shock layer q is comparable to p because of the abnormally large value of $\frac{\partial V}{\partial t}$ encountered in that region. Hence requirement (3) is satisfied. Therefore, for this particular case of steady-state plane shock, Von Neumann and Richtmyer have shown that their expression for q conforms to all the necessary restrictions.

D. STABILITY OF THE DIFFERENTIAL EQUATIONS

Our next concern will be the effect the introduction of an artificial viscosity term has on the entire system of differential equations. Before investigating the stability it should be noted that much of the computational work actually done will be omitted for the sake of brevity. Instead, reference will be made to the approach and ideas involved.

On a given solution $U(x,t)$, $V(x,t)$, etc., a small perturbation δU, δV, etc., is superimposed. Then the system will be stable if the perturbation can be kept arbitrarily small for all $t \geq T_0$ by initially choosing the perturbation at time T_0 to be sufficiently small. Therefore consider the following variations:

$$U \rightarrow U + \delta U$$

$$V \rightarrow V + \delta V$$

$$p \rightarrow p + \delta p$$

$$q \rightarrow q + \delta q$$

We then obtain our equations of first variation (i.e., higher order variations are considered to be negligible):

$$\rho_0 \frac{\partial(\delta U)}{\partial t} = - \frac{\partial(\delta p + \delta q)}{\partial x} \tag{2.48}$$

$$\frac{\partial V}{\partial t}[\gamma \delta p + (\gamma - 1)\delta q] + [\gamma p + (\gamma - 1)q] \frac{\partial(\delta V)}{\partial t} + V \frac{\partial(\delta p)}{\partial t} + \frac{\partial p}{\partial t} \delta V = 0 \tag{2.49}$$

$$\delta q = \frac{(c\Delta x)^2}{V^2} \frac{\partial U}{\partial x} \left|\frac{\partial U}{\partial x}\right| \delta V - 2 \frac{(c\Delta x)^2}{V} \left|\frac{\partial U}{\partial x}\right| \frac{\partial(\delta U)}{\partial x} \tag{2.50}$$

$$\rho_0 \frac{\partial(\delta V)}{\partial t} = \frac{\partial(\delta U)}{\partial x} \tag{2.51}$$

Equations (2.48) through (2.51) are a set of simultaneous, linear differential equations in δU, δV, δp, and δq. Their coefficients are composed of terms depending on the solution functions U, V, p, and q. Since U, V, p, and q are considered to be smooth, well-behaved functions of x and t, they will be considered to be constants in a small region. Equations (2.48) through (2.51) were combined into one equation and a separation of variable technique was employed. Using this technique possible solutions were of the form:

$$
(2.52)\qquad \begin{cases}
\delta U = \delta U_o e^{\alpha t}(\cos kx + i \sin kx) = \delta U_o e^{ikx+\alpha t} \\
\delta V = \delta V_o e^{\alpha t}(\cos kx + i \sin kx) = \delta V_o e^{ikx+\alpha t} \\
\delta p = \delta p_o e^{\alpha t}(\cos kx + i \sin kx) = \delta p_o e^{ikx+\alpha t} \\
\delta q = \delta q_o e^{\alpha t}(\cos kx + i \sin kx) = \delta q_o e^{ikx+\alpha t}
\end{cases}
$$

where δU_o, δV_o, δp_o, δq_o, k, and α are constants and k is real. Substitution of equations (2.52) into equations (2.48) through (2.51) yields the following set of simultaneous homogeneous linear equations in δU_o, δV_o, δp_o, and δq_o:

$$RS = 0 \tag{2.53}$$

where R is the following 4 x 4 matrix:

$$
R = \begin{pmatrix}
\alpha\rho_o & ik & ik & 0 \\
0 & [\frac{\partial V}{\partial t}\gamma + \alpha V] & [(\gamma-1)\frac{\partial V}{\partial t}] & [\gamma p\alpha + (\gamma-1)q\alpha + \frac{\partial p}{\partial t}] \\
[2ik\frac{(c\Delta x)^2}{V}\left|\frac{\partial U}{\partial x}\right|] & 0 & 1 & [-\frac{(c\Delta x)^2}{V^2}\frac{\partial U}{\partial x}\left|\frac{\partial U}{\partial x}\right|] \\
-ik & 0 & 0 & \alpha\rho_o
\end{pmatrix} \tag{2.54}
$$

and S is the following vector:

$$
S = \begin{pmatrix}
\delta U_o e^{ikx+\alpha t} \\
\delta p_o e^{ikx+\alpha t} \\
\delta q_o e^{ikx+\alpha t} \\
\delta V_o e^{ikx+\alpha t}
\end{pmatrix} \tag{2.55}
$$

Now if equation (2.53) is to hold for a nontrivial S, then

$$\mathrm{DET}(R) = 0 \tag{2.56}$$

The characteristic equation is:

$$(\alpha\rho_0)^2\gamma\frac{\partial V}{\partial t} + 2\frac{\rho_0\alpha}{V}\frac{\partial V}{\partial t}(kc\Delta x)^2\left|\frac{\partial U}{\partial x}\right| - \frac{1}{V^2}\frac{\partial V}{\partial t}(kc\Delta x)^2\frac{\partial U}{\partial x}\left|\frac{\partial U}{\partial x}\right| + k^2\alpha[\gamma p+(\gamma-1)q]$$

$$+ \rho_0^2\alpha^3 V + 2\rho_0\alpha^2(kc\Delta x)^2\left|\frac{\partial U}{\partial x}\right| - \frac{\alpha}{V}(kc\Delta x)^2\frac{\partial U}{\partial x}\left|\frac{\partial U}{\partial x}\right| + k^2\frac{\partial p}{\partial t} = 0 \tag{2.57}$$

Equation (2.57) establishes the relationship between α and k. By fixing k and examining the corresponding α, we can investigate the behavior of the perturbation. It should be noted that the behavior of the perturbation must be examined both in the shock regions and in the normal regions. In the shock regions all terms will be retained. In the normal regions, terms containing dissipative factors (i.e., terms containing Δx) are dropped. Now we are interested only in the α's corresponding to very large k's. Hence we retain only the dominant terms, in α and k, of equation (2.57). The dominant term in α is $\rho_0^2\alpha^3V$. The dominant term in k contains k^2 and is therefore dominated by $2\rho_0\alpha^2(kc\Delta x)^2\left|\frac{\partial U}{\partial x}\right|$, which is the dominant term in αk. Hence in the shock regions, equation (2.57) reduces to:

$$\rho_0^2\alpha^3 V + 2\rho_0\alpha^2(kc\Delta x)^2\left|\frac{\partial U}{\partial x}\right| = 0, \tag{2.58}$$

giving

$$\alpha = \frac{-2(kc\Delta x)^2\left|\frac{\partial U}{\partial x}\right|}{\rho_0 V}, \tag{2.59}$$

and in the normal regions

$$k^2\alpha\gamma p + \rho_0{}^2\alpha^3 V = 0, \tag{2.60}$$

giving

$$\alpha^2 = -\frac{k^2\gamma p}{\rho_0{}^2 V} . \tag{2.61}$$

Thus, in the normal regions our original system of differential equations is stable, and in the shock regions, the system is asymptotically stable. Von Neumann and Richtmyer also point out the terms in the equations of variation that lead to the dominant terms in equation (2.57). In the shock regions:

$$\frac{\partial(\delta U)}{\partial t} \simeq \sigma\frac{\partial^2(\delta U)}{\partial x^2} \tag{2.62}$$

where

$$\sigma = \frac{2(c\Delta x)^2}{V\rho_0}\left|\frac{\partial U}{\partial x}\right| , \tag{2.63}$$

and in the normal regions:

$$\frac{\partial^2(\delta U)}{\partial t^2} \simeq S_0{}^2\frac{\partial^2(\delta U)}{\partial x^2} \tag{2.64}$$

where

$$S_0{}^2 = \frac{\gamma p}{\rho_0{}^2 V} \tag{2.65}$$

E. FINITE DIFFERENCE SCHEME

To solve the system of differential equations several finite difference schemes could be used. The one Von Neumann and Richtmyer offer is ingeniously simple. The central differences used are skillfully staggered, taking advantage of the artificial nature of q.

Let our rectangular mesh with increments Δx and Δt, and integers $i = 0, 1, 2, \ldots, L$; $n = 0, 1, 2, \ldots$ be contrived in the following fashion:

$$U^n_{i+1/2} = U[(i+1/2)\Delta x, n\Delta t]$$

$$V^n_{i+1/2} = V[(i+1/2)\Delta x, n\Delta t] \text{ , etc.} \tag{2.67}$$

After rewriting equation (2.4) the finite difference scheme is as follows:

$$\rho_0 \frac{U^{n+1/2}_i - U^{n-1/2}_i}{\Delta t} = - \frac{p^n_{i+1/2} + q^{n-1/2}_{i+1/2} - p^n_{i-1/2}}{\Delta x} - \frac{q^{n-1/2}_{i-1/2}}{\Delta x} \tag{2.68}$$

$$\rho_0 \frac{V^{n+1}_{i+1/2} - V^n_{i+1/2}}{\Delta t} = \frac{U^{n+1/2}_{i+1} - U^{n+1/2}_i}{\Delta x} \tag{2.69}$$

$$q^{n+1/2}_{i+1/2} = - \frac{2(c\Delta x)^2 (U^{n+1/2}_{i+1} - U^{n+1/2}_i) \left| U^{n+1/2}_{i+1} - U^{n+1/2}_i \right|}{(\Delta x)^2 \; (V^n_{i+1/2} + V^{n+1}_{i+1/2})} \tag{2.70}$$

$$\left[\gamma \frac{p^{n+1}_{i+1/2} + p^n_{i+1/2}}{2} + (\gamma-1) q^{n+1/2}_{i+1/2}\right] \frac{(V^{n+1}_{i+1/2} - V^n_{i+1/2})}{\Delta t} + \frac{(V^{n+1}_{i+1/2} + V^n_{i+1/2})(p^{n+1}_{i+1/2} - p^n_{i+1/2})}{2\Delta t} = 0 \tag{2.71}$$

Since central differences were used the discretization error will be $O(\Delta x)^2$ and $O(\Delta t)^2$.

F. STABILITY OF THE FINITE DIFFERENCE SCHEME

Having shown the original system of differential equations is stable and having chosen a suitable finite difference scheme one must now show that the difference equations are in fact stable. A finite difference scheme is said to be unstable if the rounding error, introduced in approximating the numerical solution, grows exponentially with each iteration, making nonsense of our numerical data. This concept is analogous to the one for differential equations in that the rounding error corresponds to a small perturbation in the numerical solution.

Von Neumann and Richtmyer have shown equations (2.68) through (2.71) to be conditionally stable (i.e., stable only for certain combinations of Δx and Δt). Hence certain restrictions will be placed on the choice of Δx and Δt. It should be further noted that these restrictions will not be the same in shock regions as in normal regions. As before much of the computational work actually done will be omitted for brevity and clarity.

Outside shock regions the stability criteria of Von Neumann and Richtmyer is

$$\frac{\Delta t}{\Delta x} \left(\frac{\gamma p}{\rho_0^{\,2} V}\right)^{1/2} \leq 1. \tag{2.72}$$

Equation(2.72) is the usual stability criteria encountered when hydrodynamic equations of the form of equation (2.64) are approximated by central differences.

A similar analysis in the shock regions yields

$$\frac{2\sigma\Delta t}{(\Delta x)^2} \leq 1. \tag{2.73}$$

From equations (2.38), (2.39), and (2.62)

$$\sigma = sc\Delta x \left[\frac{V_i - V_f}{V}\right] \left[\frac{\gamma+1}{2}\right]^{1/2} \cos\frac{w}{w_o} \quad . \tag{2.74}$$

Equation (2.73) then becomes

$$\frac{\Delta t}{\Delta x} \leq \left(\frac{2}{\gamma+1}\right)^{1/2} \frac{\frac{J+1}{J-1} + \sin\frac{w}{w_o}}{4sc\cos\frac{w}{w_o}} \tag{2.75}$$

where

$$J = \frac{V_i}{V_f} \tag{2.76}$$

Now let

$$F\left(\frac{w}{w_o}\right) = \frac{\frac{J+1}{J-1} + \sin\frac{w}{w_o}}{\cos\frac{w}{w_o}}$$

$$= \frac{J+1}{J-1}\sec\frac{w}{w_o} + \tan\frac{w}{w_o} \quad , \tag{2.77}$$

for $-\pi/2\ w_o < w < \pi/2\ w_o$.

Then,

$$F' = \frac{J+1}{J-1}\sec\frac{w}{w_o}\tan\frac{w}{w_o} + \sec^2\frac{w}{w_o}$$

$$= \frac{J+1}{J-1}\ \frac{\sin\frac{w}{w_o}}{\cos^2\frac{w}{w_o}} + \frac{1}{\cos^2\frac{w}{w_o}}$$

$$= \sec^2\frac{w}{w_o}\left[\left(\frac{J+1}{J-1}\right)\sin\frac{w}{w_o} + 1\right] \tag{2.78}$$

Now setting $F' = 0$ implies

$$\sin \frac{w}{w_o} = -\left(\frac{J-1}{J+1}\right) , \tag{2.79}$$

since sec θ is never zero. Hence, we have obtained as a critical value

$$F = \frac{\frac{J+1}{J-1} - \frac{J-1}{J+1}}{\left[1 - \frac{(J-1)^2}{(J+1)^2}\right]^{1/2}}$$

$$= \frac{2J^{1/2}}{J-1} \tag{2.80}$$

Noting that as $\frac{w}{w_o} \to \frac{\pi}{2}$ from the left $F(\frac{w}{w_o}) \to \infty$. Since only one critical point has been found, equation (2.80) must be the minimum value for $F(\frac{w}{w_o})$ in the shock region $(-\pi/2\ w_o < w < \pi/2\ w_o)$. Consequently, equation (2.75) may be rewritten as

$$\frac{\Delta t}{\Delta x} \leq \left[\frac{2}{\gamma+1}\right]^{1/2} \left[\frac{J^{1/2}}{2sc(J-1)}\right] \tag{2.81}$$

From equation (2.21)

$$M = \left[\frac{(p_f-p_i)}{(V_f-V_i)}\right]^{1/2} . \tag{2.82}$$

Equations (2.22) and (2.6) give

$$\frac{p_f V_f}{\gamma-1} - \frac{p_i V_i}{\gamma-1} = 1/2\ (p_i+p_f)(V_i-V_f)$$

$$= 1/2\ p_i(V_i-V_f) + 1/2\ p_f(V_i-V_f), \tag{2.83}$$

so that

$$\frac{p_f V_f}{\gamma-1} - 1/2\ p_f(V_i-V_f) = \left[1/2 + \frac{1}{\gamma-1}\right]p_i V_i - \frac{p_i V_f}{2} , \tag{2.84}$$

yielding

$$\frac{2p_f V_f}{\gamma-1} - p_f\,(V_i - V_f) = [(\frac{\gamma+1}{\gamma-1})\,V_i - V_f]\;p_i\;. \qquad (2.85)$$

Then equations (2.10) and (2.21) give, after eliminating p_i,

$$S = \frac{1}{\rho_o}\left\{\frac{p_f - \left[\dfrac{\dfrac{2p_f V_f}{(\gamma-1)} - p_f(V_i - V_f)}{(\frac{\gamma+1}{\gamma-1})\,V_i - V_f}\right]}{(V_i - V_f)}\right\}^{1/2}$$

or

$$S = \frac{1}{\rho_o}\left\{[\gamma\frac{p_f}{V_f}]\;\frac{[\frac{1}{\gamma} - \frac{1}{\gamma}\,\frac{(\gamma+1)-J(\gamma-1)}{(\gamma+1)J-(\gamma-1)}]}{J-1}\right\}^{1/2}$$

$$= \frac{1}{\rho_o}\;[\frac{2}{J(\gamma+1)-(\gamma-1)}]^{1/2}\cdot[\gamma\,\frac{p_f}{V_f}]^{1/2}$$

Finally

$$S = \frac{1}{\rho_o}\;[\frac{2}{J(\gamma+1)-(\gamma-1)}]^{1/2}\;S_{of} \qquad (2.86)$$

where S_{of} is the speed of sound behind the shock, relative to x.
Now equations (2.86) and (2.81) give

$$\frac{\Delta t}{\Delta x} \le \frac{1}{2S_{of}c}\left\{\frac{[\,J-\frac{(\gamma-1)}{(\gamma+1)}]J}{(J-1)^2}\right\}^{1/2}\;. \qquad (2.87)$$

Since V_f, and hence J, is generally unknown until the problem has been solved, the term inside the radical is replaced by its minimum with respect to J:

Let

$$Q(J) = \frac{\left[\left(J - \frac{\gamma-1}{\gamma+1}\right) J\right]^{1/2}}{(J-1)} . \tag{2.88}$$

Then

$$Q' = \frac{1/2(J-1)\left[\left(J-\frac{\gamma-1}{\gamma+1}\right)J\right]^{-1/2} \cdot \left[2J-\frac{\gamma-1}{\gamma+1}\right] - \left[\left(J-\frac{\gamma-1}{\gamma+1}\right)J\right]^{1/2}}{(J-1)^2}$$

$$= \frac{1/2(J-1)\left(2J-\frac{\gamma-1}{\gamma+1}\right) - \left[J-\frac{\gamma-1}{\gamma+1}\right] J}{(J-1)^2 \left[\left(J-\frac{\gamma-1}{\gamma+1}\right) J\right]^{1/2}} . \tag{2.89}$$

Setting $Q' = 0$ gives

$$J\left[1/2\left(\frac{\gamma-1}{\gamma+1}\right) - 1\right] + 1/2\left(\frac{\gamma-1}{\gamma+1}\right) = 0 \tag{2.90}$$

so that,

$$J = \frac{\gamma-1}{\gamma+3} . \tag{2.91}$$

But

$$1 \leq J \leq \frac{\gamma+1}{\gamma-1} , \tag{2.92}$$

since $J = \frac{\gamma+1}{\gamma-1}$ corresponds to an infinitely strong shock and $J < 1$ implies that Q is imaginary. Since $\frac{\gamma-1}{\gamma+3} < 1$, there are no critical points in the interval $[1, \frac{\gamma+1}{\gamma-1}]$. Hence $J = \frac{\gamma-1}{\gamma+3}$ is the point where Q assumes its minimum since Q can be made arbitrarily large as $J \to 1$ from the right. Hence

$$Q_{min} = \frac{\left[\left(\frac{\gamma+1}{\gamma-1}\right)\left(\left(\frac{\gamma+1}{\gamma-1}\right) - 1\right)\right]^{1/2}}{\left(\frac{\gamma+1}{\gamma-1}\right) - 1}$$

$$= \gamma^{1/2} . \tag{2.93}$$

Finally substitution of (2.93) into (2.87) yields

$$\frac{\Delta t}{\Delta x} \leq \frac{\gamma^{1/2}}{2S_{of}c} \quad . \tag{2.94}$$

Equation (2.94) is then a sufficient condition for stability of the difference equations inside the shock region. It should be noted that Von Neumann and Richtmyer have ignored boundary conditions in their stability analysis. Should derivative boundary conditions enter the problem an appropriate stability analysis will have to include a discussion of the difference equations used to approximate the boundary conditions.

III. INTEGRATION OF THE EQUATIONS OF TRUE VISCOUS FLOW

An alternate method to the numerical calculation of hydrodynamic shocks is offered by Ludford, Polachek and Seeger [Ref. 5]. Their approach is similar to that of Von Neumann and Richtmyer in that their analysis examines the viscosity of the fluid. However, by replacing the fluid continuum by a particle model, they show that the true equations of viscous flow can be numerically integrated.

A. THE BASIC EQUATIONS

The equations of one dimensional flow of a perfect viscous compressible fluid may be written

$$\rho \frac{DU}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \sigma}{\partial x}, \tag{3.1}$$

$$\frac{D\rho}{Dt} = -\rho \frac{\partial U}{\partial x} \tag{3.2}$$

$$\sigma \frac{\partial U}{\partial x} = \rho T \frac{DS}{Dt} \tag{3.3}$$

$$p = \rho \overline{R} T \; ; \quad \overline{R} = C_v(\gamma - 1), \tag{3.4}$$

where

$$\sigma = \frac{4}{3}\mu \frac{\partial U}{\partial x}, \tag{3.5}$$

$$S = C_v \log(p/\rho^{\gamma}). \tag{3.6}$$

T, σ, S, and u are the temperature, viscous stress, entropy per unit mass, and coefficient of viscosity respectively. D represents the rate of change of the quantity written after it, if we move with the gas particle (Lagrangean viewpoint). All other quantities were defined previously. Now p may be eliminated from eqn. (3.3) by use of eqns.

(3.4), (3.6) and (3.2) in the following manner:

$$\sigma \frac{\partial U}{\partial x} = \rho T \frac{DS}{Dt} = \frac{p}{\overline{R}T} (T) \frac{DS}{Dt} ,$$

giving

$$\sigma \frac{\partial U}{\partial x} = \frac{p}{C_v(\gamma-1)} \frac{DS}{Dt} . \tag{3.7}$$

Since

$$\frac{DS}{Dt} = C_v \frac{D}{Dt} \{\log(p/\rho^\gamma)\} = C_v\{ \frac{1}{p} \frac{Dp}{Dt} - \frac{\gamma}{\rho} \frac{D\rho}{Dt}\} \tag{3.8}$$

we have

$$\sigma \frac{\partial U}{\partial x} = \frac{p}{\gamma-1} \{ \frac{1}{p} \frac{Dp}{Dt} - \frac{\gamma}{\rho} \frac{D\rho}{Dt} \} . \tag{3.9}$$

Then

$$\sigma \frac{\partial U}{\partial x} - \frac{1}{(\gamma-1)} (\frac{Dp}{Dt}) = (\frac{\gamma}{\gamma-1}) p \frac{\partial U}{\partial x} \tag{3.10}$$

finally giving

$$\frac{Dp}{Dt} = \frac{\partial U}{\partial x} [\sigma(\gamma-1) - \gamma p] \tag{3.11}$$

Note that eqn. (3.11) is independent of ρ.

Now Ludford et al. approximate the motion of the fluid by the motion of a system of particles. They consider flow inside a tube of unit cross-section. The tube is considered to contain (N+2) particles, each of mass m, and each moving along a fixed straight line. Each particle then represents that mass m of gas which initially had the particle at its center (see figure 3). Hence the forces encountered within the original gas may be approximated by the interaction of forces between the particles.

If x_n is the x coordinate of the n^{th} particle, and a dot represents differentiation with respect to time, the viscous interaction between the n^{th} and $(n+1)^{th}$ particles is given by

$$\sigma_{n+\frac{1}{2}} = \frac{4}{3}\mu\left[\frac{\dot{x}_{n+1} - \dot{x}_n}{x_{n+1} - x_n}\right] , \tag{3.12}$$

as a result of eqn. (3.5). Now from eqn. (3.1)

$$\rho\ddot{x}_n = \frac{-(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}})}{(x_{n+\frac{1}{2}} - x_{n-\frac{1}{2}})} + \frac{(\sigma_{n+\frac{1}{2}} - \sigma_{n-\frac{1}{2}})}{(x_{n+\frac{1}{2}} - x_{n-\frac{1}{2}})} , \tag{3.13}$$

giving

$$m\ddot{x} = -(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}}) + (\sigma_{n+\frac{1}{2}} - \sigma_{n-\frac{1}{2}}) , \tag{3.14}$$

since

$$m = \rho(x_{n+\frac{1}{2}} - x_{n-\frac{1}{2}})(1). \tag{3.15}$$

The pressure interaction between the n^{th} and $(n+1)^{th}$ particles is given by

$$\dot{p}_{n+\frac{1}{2}} = \frac{\dot{x}_{n+1} - \dot{x}_n}{x_{n+1} - x_n}\left[(\gamma-1)\sigma_{n+\frac{1}{2}} - \gamma p_{n+\frac{1}{2}}\right]. \tag{3.16}$$

If the constants u and γ are those of the original gas equation then eqn. (3.2) is automatically satisfied. Hence eqns. (3.12) through (3.16) are sufficient to characterize the particle model.

Before choosing a finite difference scheme, Ludford, Polachek, and Seeger nondimensionalize eqns. (3.12), (3.13), (3.14), and (3.16) by the following transformations:

$$\begin{aligned} p &= p_o P = \frac{1}{\gamma}\rho_o a_o{}^2 P \\ \sigma &= \frac{\mu}{\ell} V\Sigma^* \\ x &= \ell X \\ t &= \frac{\ell T^*}{V} \\ a_o{}^2 &= \frac{\gamma p_o}{\rho_o} \end{aligned} \tag{3.17}$$

giving in place of the original equations of particle motion,

$$\Sigma_{n+\frac{1}{2}} = \frac{X'_{n+1} - X'_n}{X_{n+1} - X_n}$$

$$X_n'' = -(P_{n+\frac{1}{2}} - P_{n-\frac{1}{2}}) + A(\Sigma_{n+\frac{1}{2}} - \Sigma_{n-\frac{1}{2}}) \qquad (3.18)$$

$$P'_{n+\frac{1}{2}} = \frac{X'_{n+1} - X'_n}{X_{n+1} - X_n} \; [(\gamma-1)A\Sigma_{n+\frac{1}{2}} - \gamma P_{n+\frac{1}{2}}] \; ,$$

where primes denote differentiation with respect to T, and

$$A = \left(\frac{4}{3}\right) \frac{V\mu}{a_o V\ell\rho_o} \; [\gamma(N+1)]^{1/2} \qquad (3.19)$$

B. FINITE DIFFERENCE EQUATIONS

The finite difference scheme chosen to represent eqns. (3.18) is as follows:

$$\Sigma^{m+\frac{1}{2}}_{n+\frac{1}{2}} = \frac{2}{\Delta T}\left[\frac{(X^{m+1}_{n+1} - X^m_{n+1}) - (X^{m+1}_n - X^m_n)}{(X^{m+\frac{1}{2}}_{n+1} + X^m_{n+1}) - (X^{m+1}_n + X^m_n)}\right]$$

$$\Sigma^m_{n+\frac{1}{2}} = \frac{1}{2}\left[\Sigma^{m+\frac{1}{2}}_{n+\frac{1}{2}} + \Sigma^{m-\frac{1}{2}}_{n+\frac{1}{2}}\right] \qquad (3.20)$$

$$X^{m+1}_n - 2X^m_n + X^{m-1}_n = (\Delta T)^2[-(P^m_{n+\frac{1}{2}} - P^m_{n-\frac{1}{2}}) + A(\Sigma^m_{n+\frac{1}{2}} - \Sigma^m_{n-\frac{1}{2}})]$$

$$P^{m+1}_{n+\frac{1}{2}} - P^m_{n+\frac{1}{2}} = [\Delta T\Sigma^{m+\frac{1}{2}}_{n+\frac{1}{2}}][(\gamma-1)A\Sigma^{m+\frac{1}{2}}_{n+\frac{1}{2}} - \frac{\gamma}{2}(P^m_{n+\frac{1}{2}} + P^{m+1}_{n+\frac{1}{2}})]$$

In the system, the quantity $P^m_{n+\frac{1}{2}}$ is the pressure between the n^{th} and (n+1)th particles at time $T = m\Delta T$. It should be noted that the above system cannot be solved explicitly since the superscripts of Σ are staggered. Hence eqns. (3.20) must be solved by an iterative procedure at every $\Delta T/2$ time increment.

The stability analysis of the finite difference equations is quite similar to that of Von Neumann and Richtmyer. Equations (3.20) are numerically stable if, in the normal regions,

$$\Delta T \leq \left[\frac{(X_n - X_{n-1})}{\gamma P}\right]^{1/2} . \tag{3.21}$$

The system is unconditionally stable in shock regions.

The approach taken by Ludford, Polachek, and Seeger has some advantages over that of Von Neumann and Richtmyer. First, in the particle model approach, the stability requirements are somewhat less stringent. Secondly, this approach gives a fairly accurate representation of the behavior of the fluid in the shock regions, whereas the artificial viscosity approach does not represent the physical system in the shock layers. However the approach by Ludford et al. does have a major disadvantage. If the fluid under investigations has a very low viscosity an inconveniently fine mesh will have to be used in order that the finite difference approximations are accurate beyond the shock front. Otherwise ΔX will be larger than the thickness of the shock wave. It should be recalled that in the artificial viscosity approach q automatically adjusted to the shock thickness.

IV. THE METHOD OF LAX

It was stated earlier that approaches to the numerical calculation of compressible fluid flow generally fall into two categories. The first category consisted of methods which examined the viscosity of the fluid. The motivation for this approach was that when a nonlinear term, multiplied by a small coefficient, is introduced into a differential equation, it may produce large changes in the behavior of the solution. The second category consists of methods which examine the basic equations of nonviscous flow, while allowing discontinuous solutions. An ingenious method developed by Peter Lax [Ref. 4] belongs to the second category.

It is evident that the approaches discussed so far are heuristic in nature and lack theoretical preliminaries. Lax succeeded in making several theoretical observations which could tie together the various methods discussed in this paper. Unfortunately several of his observations have been proven only for particular cases.

Observing that all nonlinear systems of fluid dynamics satisfy certain "conservation laws", Lax states that any hydrodynamic system can generally be brought into the form

$$U_t + F_x + B = 0, \qquad (4.1)$$

where U is a column vector of unknown functions, F is a column vector such that $F = F(x,t,U)$, and B is a vector coefficient. U is said to be a weak solution of eqn. (4.1) with initial value Φ if the integral relation

$$\int_0^\infty \int_{-\infty}^\infty \{W_t U + W_x F - WB\}\, dxdt + \int_{-\infty}^\infty W(x,0)\Phi(x)dx = 0 \qquad (4.2)$$

holds for every test vector W which has continuous first derivatives and vanishes outside of some bounded region in the x,t-plane. Equation (4.2) is obtained by multiplying eqn. (4.1) by W and then integrating by parts as follows:

$$0 = \int_0^\infty \int_{-\infty}^\infty \{WU_t + WF_x + WB\} \; dxdt$$

$$= \int_{-\infty}^\infty [WU]_0^\infty dx - \int_0^\infty \int_{-\infty}^\infty UW_t dxdt + \int_0^\infty [WF]_{-\infty}^\infty \; dt$$

$$- \int_0^\infty \int_{-\infty}^\infty FW_x dxdt + \int_0^\infty \int_{-\infty}^\infty WBdxdt \tag{4.3}$$

$$= I_1 - I_2 + I_3 + I_5 \; .$$

Now $I_3 = 0$ since W vanishes outside some bounded region. Similarly

$$I_1 = - \int_{-\infty}^\infty W(x,0)\Phi(x)dx. \tag{4.4}$$

Hence we obtain eqn. (4.2).

Clearly, the only requirement for U to be a weak solution is that it be continuous almost everywhere so that the (Riemann) integrals in eqn. (5.2) exist. Hence weak solutions need not be differentiable. The motivation for developing the notion of weak solutions is that in physical systems we are concerned with discontinuous functions that satisfy eqn. (4.1) almost everywhere.

The Rankine-Hugoniot equations now have an interesting interpretation in terms of weak solutions; if U_1 and U_2 are two genuine solutions of eqn. (4.1) whose domains in the x,t-plane are separated by a smooth curve, the two taken together will constitute a weak solution if and only if the slope m of the separating curve and the value of U_1 and U_2 on the curve satisfy

$$\frac{1}{m}(U_1 - U_2) = F(U_1) - F(U_2) \tag{4.5}$$

Equation (4.5) corresponds precisely to eqns. (2.21) and (2.22).

Having generalized the concept of a solution to a differential equation, we might expect some of the properties of the original concept to be lost. For instance, initial values do not in general determine a unique weak solution. This property raises immediate difficulties. In nature nonviscous flow is described by a unique weak solution to the hydrodynamic equations, given an initial vector. Hence if our mathematical model is a meaningful one, there must be some other principle that uniquely defines a weak solution. Lax offers some possibilities, the most likely being that the weak solutions occuring in nature are limits of viscous flows. It should be possible then to determine some relationship between weak solutions and solutions to the viscous flow problem.

THEOREM:

Consider the nonlinear parabolic system

$$U_t + F_x + B = \lambda U_{xx} \tag{4.6}$$

with initial vector $U(x,0) = \Phi(x)$. Here λUxx corresponds to a viscosity factor. Given that the strong limit (as $\lambda \to 0$) of the net of solution functions $U_\lambda(x,t)$ exists and is equal to $U(x,t)$, then $U(x,t)$ is a weak solution of eqn. (4.1).

PROOF:

Consider an arbitrary twice differentiable test vector W. Multiplying eqn. (4.6) by W and integrating by parts yields

$$\int_0^\infty \int_{-\infty}^\infty \{W_t U_\lambda + W_x F(U_\lambda) - WB\}\, dxdt + \int_{-\infty}^\infty W(x,0)\Phi(x)dx$$

$$= -\lambda \int_0^t \int_{-\infty}^\infty W_x U_\lambda \, dxdt \tag{4.7}$$

Now keeping Φ and W fixed, and letting $\lambda \to 0$, the left side of eqn. (4.7) approaches the left side of eqn. (4.2), and the right side of eqn. (4.7) approaches zero. Hence $U(x,t)$ satisfies eqn. (4.2) for all twice differentiable test functions. Note that U had to be a strong limit of U_λ (i.e. $\iint |U_\lambda - U| \to 0$ over any bounded region in the x,t-plane) in order that $F(U_\lambda) \to F(U)$. If $U_\lambda \to U$ only in the weak sense (i.e. $U_\lambda(x,t)$ $U(x,t) \forall (x,t)$)there is no quarantee that $F(U_\lambda)$ would converge to $F(U)$. Now it must be shown that eqn. (4.2) is satisfied for any arbitrary test vector W which is continuously once differentiable. Lax states that since U satisfies eqn. (4.2) for all twice differentiable test vectors W, à fortiore, U satisfies eqn. (4.2) for all once differentiable test vectors.[1] This author disagrees with this argument since the class of once differentiable functions is larger than the class of twice differentiable functions. Though possibly true, the statement requires proof. Several approaches were attempted and proved to be unsuccessful. One apparent way to eliminate the difficulty is to restrict our attention to just twice differentiable test vectors in our original definition of a weak solution.

Having shown for a particular case, that the limit of viscous flow (as $\lambda \to 0$) is a weak solution, Lax conjectures that all viscosity methods

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[1] Lax, P. D., "Weak Solutions of Nonlinear Hyperbolic Equations and Their Numerical Computation," Communications on Pure and Applied Mathematics, v. 7, p. 163, 1954.

should converge to the same weak solution. However he proposes a different limiting process, namely a special finite difference scheme, which is independent of the viscosity of the fluid. This approach has a significant advantage over the viscosity approach in that it accurately represents the actual behavior of the fluid in every region of the flow. Lax's method replaces the differentiations by finite-difference operations according to the scheme

$$f_x(x,t) \rightarrow \frac{1}{2\Delta x}[f(x+\Delta x,t) - f(x-\Delta x,t)]$$
$$f_t(x,t) \rightarrow \frac{1}{2\Delta t}[f(x,t+\Delta t) - \frac{f(x+\Delta x,t) - f(x-\Delta x,t)}{2}] \tag{4.8}$$

Finally, substantial numerical evidence supports Lax's conjecture that when the above finite difference scheme is applied to any single homogeneous first order conservation law

$$u_t + [f(u)]_x = 0 \qquad f'' < 0 \tag{4.9}$$

with $U(x,0) = \phi(x)$, the solution $U(x,t,\Delta x)$ approaches the same weak solution generated by the viscosity methods. The solution is given by

$$u(x,t) = g\left(\frac{x-y_o}{t}\right) \tag{4.10}$$

where $y_o = y_o(x,t)$ maximizes

$$\int_0^y \phi(s)ds + tG\left(\frac{x-y}{t}\right) \tag{4.11}$$

and

$$f'[g(s)] = s \quad ; \quad G'(s) = g \tag{4.12}$$

In most physical situations $U(x,t)$ is a piecewise differentiable function. Then it is an easy matter to show $U(x,t)$ is a weak solution to eqn. (4.9) since verification of eqn. (4.2) may be avoided. All that must be shown is that U(x,t) satisfies eqn. (4.9), wherever it has well-defined first derivatives. Since the first derivatives of U(x,t) are undefined at most on a set of measure zero, and U(x,t) satisfies eqn. (4.9) everywhere else, U(x,t) must by necessity satisfy the integral relation (4.2) everywhere. Hence (4.10) is a weak solution of eqn. (4.9). The integral eqn. (4.11) is necessary to uniquely define the weak solution.

V. CONCLUSIONS

The original intention of this paper was to analyze difficulties in a numerical system called "PUFF" [Ref. 2]. "PUFF" is an attempt to numerically compute the reaction of a multi-layered medium to violent shocks. The Puff code employs Von Neumann and Richtmyer's artificial viscosity term. It has been shown that there were very large discrepancies between the "PUFF" solution and the classical solution inside the shock regions [Ref. 1]. The reason for these errors should now be apparent. Von Neumann and Richtmyer's artificial viscosity term had to comply with certain requirements. However these requirements were to a large degree independent of actual physical considerations and, hence, were not sufficiently restrictive. Consequently, in shock regions where viscosity is significant, the artificial term may not accurately describe the actual fluid flow. This difficulty becomes critical in a multi-layered medium since the number of shock waves is increased when the original shock reflects from several boundaries.

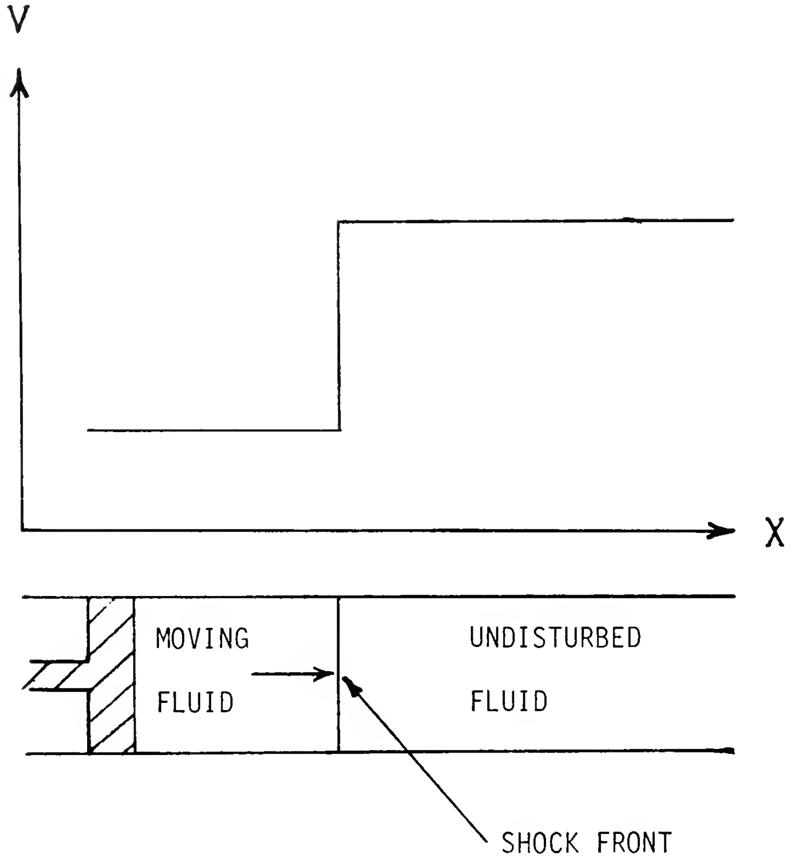

FIGURE 1 [Ref. 7]

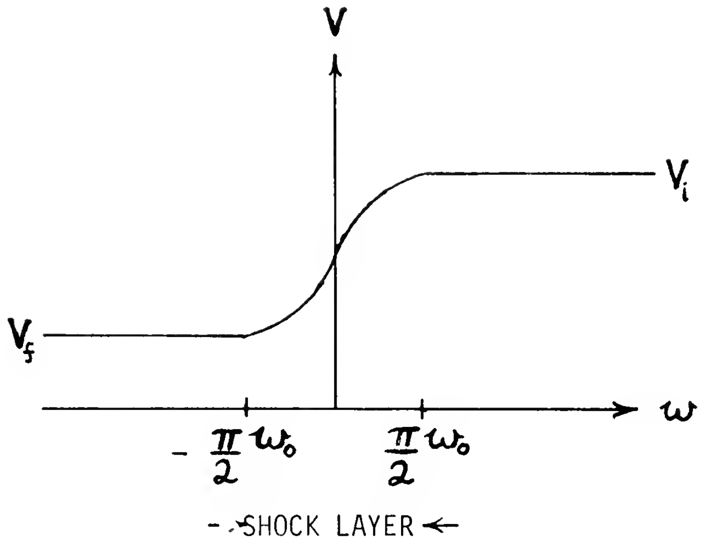

FIGURE 2 [Ref. 7]

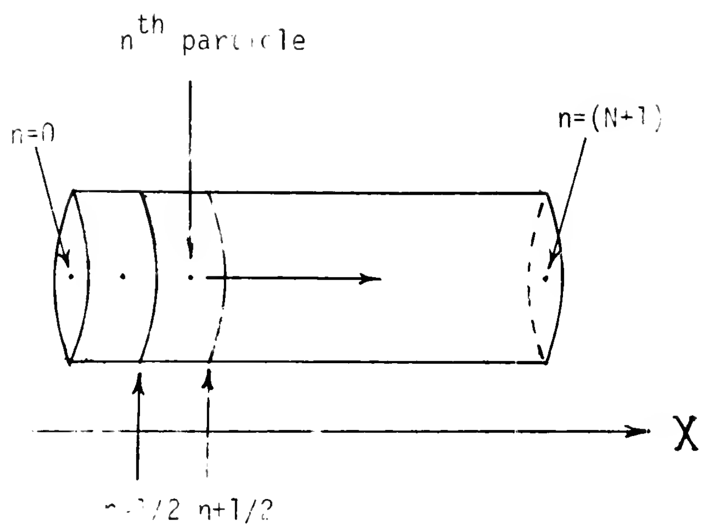

FIGURE 3

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13. ABSTRACT

Several numerical methods used in the calculation of hydrodynamic shocks were investigated. Particular attention was given to the artificial viscosity approach of Von Neumann and Richtmyer and its application to the "PUFF" numerical scheme. The particle model approach of Ludford, Polachek and Seeger, and the method of Lax were also considered.

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